Lecture 1/2 — Classical Function Spaces

Spaces of Continuous Functions

Lemma 1. *Let $I = (a, b)$ with $a, b = \pm\infty$ and let $f : I \mapsto \mathbb{C}$. f is continuous if and only if*

$$f(x) = f(c) + o(1) \qquad as \qquad |x - c| \to 0.$$

proof. $f(x) - f(c) = o(1)$ as $|x - c| \to 0 \;\Leftrightarrow \forall\epsilon > 0,\; \exists\delta > 0\; s.t$

$$0 \le f(x) - f(c) < \epsilon \text{ whenever } |x - c| < \delta.$$

$\square$

Suppose $J \subseteq I$. The set $C(J) := \{\, f : \; f \text{ is continuous at } c, \; \forall c \in J\}$ has the following properties:
1) Linear space with a commutative algebra under multiplication.
2) Stable under composition.
3) $\|f\|_{\infty,J} = \sup\limits_{x\in J}|f(x)|$. Norms may take infinite values.

If $K \subset I$ is compact, $\quad C(K) = BC(K) = C_b(K)$ where

$$BC(K) := \{\, f \in C(J) : \; \|f\|_{\infty,J} < \infty\}$$

If K is not compact then $BC(J) \subset C(J)$.

$BC(K)$ is closed under uniform convergence. This allows us to define $C(J)$ as a Banach space under the $\infty-$norm. Furthermore,

$$\|fg\|_\infty \le \|f\|_\infty \|g\|_\infty$$

$\implies$ defines a Banach Algebra.

Definition 1. *Let $\{f_k\}$ be a convergent sequence in $C(I)$. We say $f_k \to f$ locally uniformly if and only if*

$$\forall x \in I, \exists nbhd\ N \ni x\ s.t.\ \|f_k - f\|_{\infty,N} \to 0.$$

FIG 1.1.

We see from Fig 1.1 that the pulse goes to the right continuously. The sequence of functions clearly do not converge uniformly to 0 over I, however it does in a neighbourhood of x

Theorem 1. *We have local uniform convergence if and only if Compactly convergent.*

Definition 2. *f is said to be Holder Continuous at c if and only if*

$$f(x) = f(c) + O(|x - c|^\alpha).$$

When $\alpha = 1$ we say f is Lipschitz continuous.

Definition 3. *We define a norm by evaluating $\|f\|_{C^{0,\alpha}(J)} = \sup_{x,y\in J} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$ and defining its norm:*

$$\|f\|_{C^{0,\alpha}(J)} = \|f\|_{\alpha,J} + |f|_{C^{0,\alpha}(J)}.$$

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Spaces of Differentiable Functions

Examples

$$C^k(J) := \{\ f \in C^{K-1}(J) :\ f' \in C(J)\}, \qquad C^\infty(J) := \bigcap_{k=0}^{\infty} C^k(J)$$

Definition 4. *We define a norm for k-differentiable function to be*

$$\|f\|_{C^k(J)} := \sum_{j=0}^{k} \|f^{(j)}\|_{\infty,J}.$$

Definition 5. *A function $f : \Omega \mapsto \mathbb{C},\ \Omega \subseteq \mathbb{C}$, is said to be analytic at $c \in \Omega$ if and only if*

$$(*) \qquad f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$$

for $|z-c| < r$ for some $r > 0$.

Theorem 2 (Convergence Thrm Power Series)**.** *For any power there exists a uniquely determined number R with the following properties:*
(a) The series converges for any z s.t $|z-c| < R$.
(b) The series diverges for any z s.t $|z-c| > R$.

proof. Write $R := \sup\{r \geq 0 :\ \sup_n |a_n| r^n < \infty\}$. Clearly if whenever $|z-c| > R$ the series $(*)$ is divergent. Consider $0 < \rho < R$ we have $|a_n|\rho^n \leq |a_n|R^n < \infty$ so set $M = |a_n|\rho^n$. Therefore

$$\sum_{n=0}^{\infty} |a_n||z-c|^n = \sum |a_n|\rho^n \frac{|z-c|^n}{\rho^n} \leq M \sum (\frac{|z-c|}{\rho})^n < \infty,$$

by geometric progression since $(= \frac{|z-c|}{\rho} < 1$ by our choice. □

$$C^\omega(\Omega) = \{\ f \in C^\infty :\ f \text{ analytic at every } c \in \Omega\}$$

Suppose

$$a_n = \frac{f^{(n)}(c)}{n!} \qquad \textit{with} \qquad |a_n| r^n \leq M,$$

then

$$|f^{(n)}(c)| \leq \frac{Mn!}{r^n} \Leftrightarrow f \in C^\omega(c) = G^1_{M,r}(c).$$